MATH 1700: SECTION 10.3: GRAPHS OF SINE AND COSINE

DEFINING SINE AND COSINE OF REAL NUMBERS:

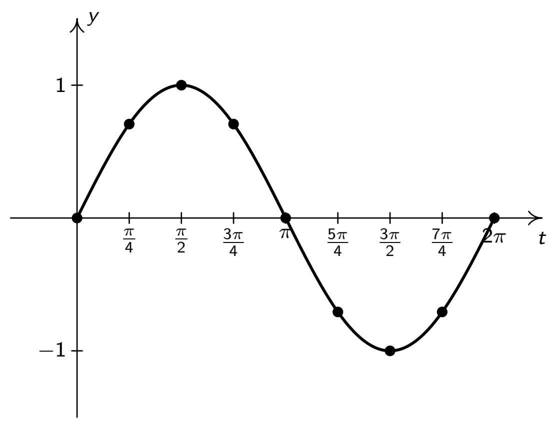
Recall that we can identify a real number t with an oriented angle θ measuring t radians and define $\sin(t) = \sin(\theta)$ and $\cos(t) = \cos(\theta)$. Since every real number can be identified with one and only one angle θ, and each angle θ is matched with one and only one value each for sine and cosine, we get two functions $f(t) = \sin(t)$ and $g(t) = \cos(t)$ with domains $(-\infty, \infty)$. Since the real number line, when wrapped around the Unit Circle, completely covers the circle, we can be assured that every point on the Unit Circle corresponds to at least one real number. Hence, the range of $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are both $[-1, 1]$.

DOMAIN AND RANGE OF THE SINE AND COSINE FUNCTIONS:

- The function $f(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $g(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$

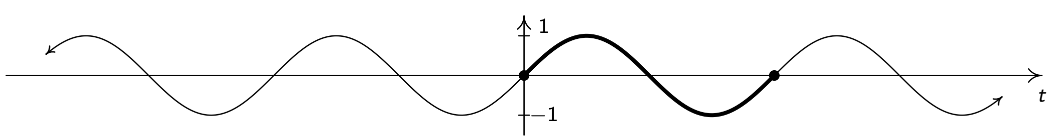
GRAPHING SINE:

t	$\sin(t)$	$(t, \sin(t))$
0	0	$(0,0)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, \frac{\sqrt{2}}{2}\right)$
π	0	$(\pi, 0)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
2π	0	$(2\pi, 0)$



Graphing $y = \sin(t)$.

If we plot additional points, we soon find that the graph repeats itself. In fact, since the circumference of the Unit Circle is 2π, we expect the function to repeat itself every 2π units. Below is a more accurately scaled graph highlighting the portion we had already graphed above. The graph is often described as having a 'wavelike' nature and is sometimes called a **sine wave** or, more technically, a **sinusoid**.



A more accurately scaled graph of $f(t) = \sin(t)$.

Note that by copying the highlighted portion of the graph and pasting it end-to-end, we obtain the entire graph of $f(t) = \sin(t)$. We give this 'repeating' property a name.

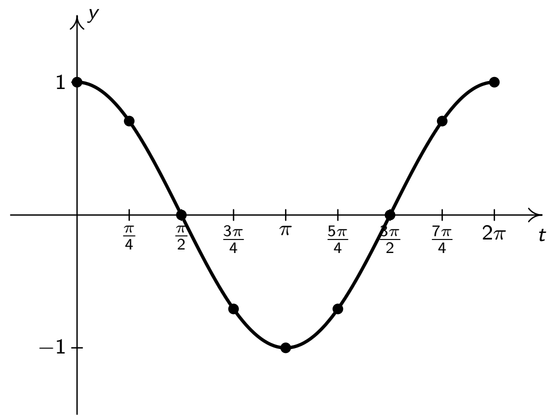
PERIODIC FUNCTIONS: A function f is said to be **periodic** if there is a real number c so that $f(t+c) = f(t)$ for all real numbers t in the domain of f. The smallest positive number p for which $f(t+p) = f(t)$ for all real numbers t in the domain of f, if it exists, is called the **period** of f.

One can show that the period of $f(t) = \sin(t)$ is 2π. This involves showing that not only is $\sin(t + 2\pi) = \sin(t)$ for all real numbers, t, but, moreover if $\sin(t + p) = \sin(t)$, then $p \geq 2\pi$. Having period 2π essentially means that we can completely understand everything about the function $f(t) = \sin(t)$ by studying *one* interval of length 2π, say $[0, 2\pi]$. For this reason, when graphing sine (and cosine) functions, we typically restrict our attention to graphing these functions over the course of one period to produce one **cycle** of the graph.

Not surprisingly, the graph of $g(t) = \cos(t)$ exhibits similar behavior as $f(t) = \sin(t)$ as seen below.

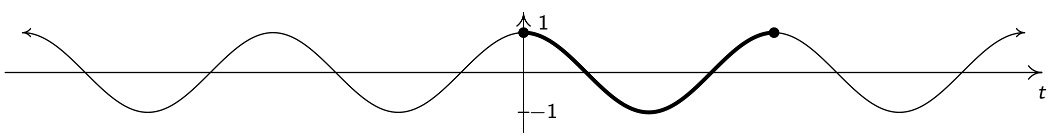
GRAPHING COSINE:

t	$\cos(t)$	$(t, \cos(t))$
0	1	$(0, 1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
π	-1	$(\pi, -1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, \frac{\sqrt{2}}{2}\right)$
2π	1	$(2\pi, 1)$



Graphing $y = \cos(t)$.

Like $f(t) = \sin(t)$, $g(t) = \cos(t)$ is a wavelike curve with period 2π. Moreover, the graphs of the sine and cosine functions have the same shape - differing only in what appears to be a horizontal shift ...



The graph of $g(t) = \cos(t)$.

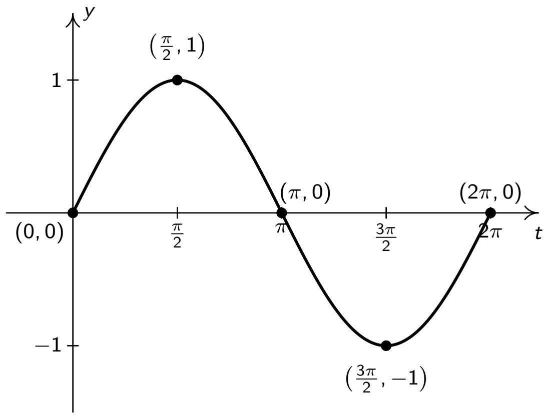
PROPERTIES OF THE SINE AND COSINE FUNCTIONS:

- The function $f(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is odd
  - has period 2π
- The function $g(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is even
  - has period 2π

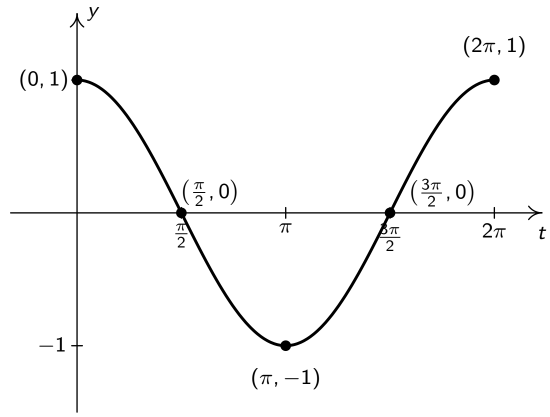
- Conversion formulas:[1] $\sin\left(t + \frac{\pi}{2}\right) = \cos(t)$ and $\cos\left(t - \frac{\pi}{2}\right) = \sin(t)$

[1] We explain these in more detail in the text.

THE FUNDAMENTAL CYCLE FOR SINE AND COSINE:



The 'fundamental cycle' of $y = \sin(t)$.



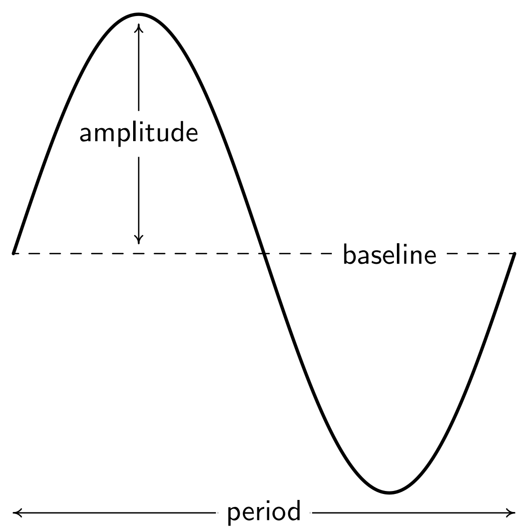
The 'fundamental cycle' of $y = \cos(t)$.

EXAMPLE 1: Use transformations to graph one cycle of each graph below. State the period of each.

1. $f(t) = 3\sin(2t)$
2. $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$

SINUSOIDS:

As previously mentioned, the curves graphed in Example 1 are examples of sinusoids. We graph one cycle of a generic sinusoid below. Sinusoids can be characterized by four properties which we define and discuss below.



We have already discussed the period of a sinusoid. If we think of t as measuring time, the period is how long it takes for the sinusoid to complete one cycle and is usually represented by the letter T. The standard period of both $\sin(t)$ and $\cos(t)$ is 2π, but horizontal scalings will change this.

The **phase shift** of the sinusoid is the horizontal shift. Again, thinking of t as time, the phase shift of a sinusoid can be thought of as when the sinusoid 'starts' as compared to $t = 0$. Assuming there are no reflections across the y-axis, we can determine the phase shift of a sinusoid by finding where the value $t = 0$ on the graph of $y = \sin(t)$ or $y = \cos(t)$ is mapped to under the transformations.

The vertical shift of a sinusoid determines the new 'baseline' of the sinusoid. Thanks to symmetry, the vertical shift can always be found by averaging the maximum and minimum values of the sinusoid.

The **amplitude** of the sinusoid is a measure of how 'tall' the wave is. Said differently, the amplitude measures how much the curve gets displaced from its 'baseline. ' The amplitude of the standard cosine and sine functions is 1, but vertical scalings can alter this.

SINUSOID CHARACTERISTICS: The graph of $S(t) = A\sin(\omega t + \phi) + B$ is called a **sinusoid.**

For $\omega > 0$, the graphs of

$$S(t) = A\sin(\omega t + \phi) + B \quad \text{and} \quad C(t) = A\cos(\omega t + \phi) + B$$

- have period $T = \dfrac{2\pi}{\omega}$
- have phase shift $-\dfrac{\phi}{\omega}$
- have amplitude $|A|$
- have vertical shift or 'baseline' B

NOTE: Since $\sin(-\theta) = -\sin(\theta)$, we always (re)-write $S(t)$ so $\omega > 0$.

The parameter ω mentioned above is called the **angular frequency**, or more simply, the **frequency** of the sinusoid and is the number of cycles the sinusoid completes over an interval of length 2π. That is, ω measures how 'frequently' the sinusoid repeats over an interval of length 2π. If t represents time, ω as represents how fast the sinusoid is being generated in terms of *radians* per unit time. In essence, it is the *angular speed* of the curve.

A quantity closely related to the angular frequency of the sinusoid is the **ordinary frequency** of the sinusoid, usually denoted f. The ordinary frequency of a sinusoid measures the number of cycles the sinusoid completes over an interval of length 1. Since the period, T represents the length of the interval required for a sinusoid to make one complete cycle, we have $f = \frac{1}{T}$. Once again, if t represents time, the ordinary frequency measures how fast the sinusoid is being generated in terms of *complete cycles* per unit time.

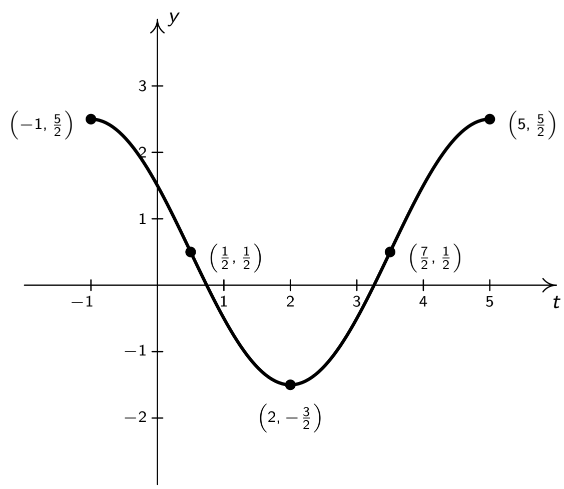
Note that since $T = \frac{2\pi}{\omega}$, $f = \frac{1}{T} = \frac{\omega}{2\pi}$. Rewriting, we get $\omega = 2\pi f$. To understand this equation in terms of units, recall 1 complete cycle (revolution) around the Unit Circle counts for 2π radians. Hence, to get from f, measured in cycles per unit time, to ω, measured in radians per unit time, we need to multiply by 2π.

Last, but not least, the quantity ϕ is often called the **phase** or **phase angle** of the sinusoid. The phase angle is important in describing waves in fields such as physics and electronics since it represents the offset of waves with the same frequency, e.g., $y = A\sin(\omega t)$ and $y = A\sin(\omega t + \phi)$. When *graphing* sinusoids, however, we focus our attention on the horizontal shift induced by ϕ, $-\frac{\phi}{\omega}$.

EXAMPLE 2: Determine the frequency, period, phase shift, amplitude, and vertical shift of each of the following functions and use this information to graph one cycle of each function.

1. $f(t) = 3\cos\left(\frac{\pi t - \pi}{2}\right) + 1$
2. $g(t) = \frac{1}{2}\sin(\pi - 2t) + \frac{3}{2}$

EXAMPLE 3: Below is the graph of one complete cycle of a sinusoid $y = f(t)$.



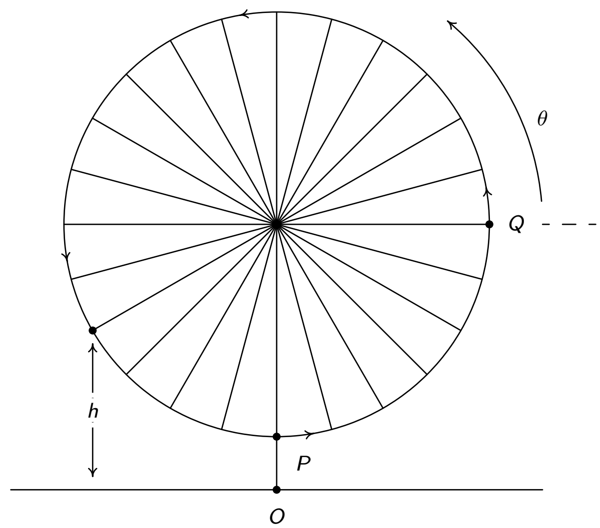
One cycle of $y = f(t)$.

1. Write $f(t)$ in the form $C(t) = A\cos(\omega t + \phi) + B$, for $\omega > 0$.

2. Write $f(t)$ in the form $S(t) = A\sin(\omega t + \phi) + B$, for $\omega > 0$.

APPLICATIONS OF SINUSOIDS:

EXAMPLE 4: The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height 136 feet. It completes two revolutions in 2 minutes and 7 seconds. Assuming that the riders are at the edge of the circle, find a sinusoid which describes the height of the passengers above the ground t seconds after they pass the point on the wheel closest to the ground.



EXAMPLE 5: According to the U.S. Naval Observatory website, the number of hours H of daylight that Fairbanks, Alaska received on the 21st day of the nth month of 2009 is given below. Here $t = 1$ represents January 21, 2009, $t = 2$ represents February 21, 2009, and so on.

Month Number	1	2	3	4	5	6	7	8	9	10	11	12
Hours of Daylight	5.8	9.3	12.4	15.9	19.4	21.8	19.4	15.6	12.4	9.1	5.6	3.3

1. Find a sinusoid which models these data and graph your answer along with the data.

2. Compare your answer to part 1 to one obtained using the regression feature of a graphing utility.

HOMEWORK: Section 10.3: Page 860: 1 - 23 all, 25 - 31 odd, 30 - 32 all